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Axial and gauge anomalies in the field–antifield quantization of the generalized Schwinger model

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Abstract. In the generalized Schwinger model the vector and axial vector currents are linearly coupled, with arbitrary coefficients, to the gauge connection. Therefore it represents an interesting example of a theory where both gauge anomalies and anomalous divergences of global currents show up in general. We derive results for these two kinds of quantum corrections inside the field–antifield framework.

An anomaly corresponds to the violation, at the quantum level, of some classical symmetry. In a path integral quantization scheme, this is reflected in the non-trivial behaviour of the path integral measure with respect to the corresponding transformation. It is important to distinguish between anomalies in gauge (local) and in global symmetries. Gauge invariance has the important consequence of implying a set of relations among the Greens functions of a field theory, the so called Ward identities, that play a crucial role, for example, in the study of renormalizability. A quantum obstruction to this kind of symmetry thus corresponds to a failure in the process of quantization of a classically gauge-invariant theory.

Concerning the case of global symmetries, the Noether theorem implies, at the classical level, that to each global invariance corresponds an associated conserved current. Quantum effects can change this picture in such a way that the expectation value of the divergence of a classical conserved current can assume a non-vanishing value. Even some non-vanishing divergences, that so do not correspond to true symmetries of the classical action, can have their expectation values modified by quantum effects. This can also be considered as a global anomaly, as it happens with the covariant divergence of the non-Abelian axial current in QCD_4. It is important to note that these global anomalous behaviours show up also in field theories that are normally referred in the literature as 'non-anomalous' (in the sense that they do not have gauge anomalies). In QCD_4, for instance, adopting a regularization that considers the vector gauge symmetry as a preferred one, the gauge invariance is not broken at quantum level but the expectation value of the axial current gets anomalous contributions [1, 2].

The Lagrangian quantization of Batalin and Vilkovisky (BV) [3–5], also known as the field–antifield formalism, is considered to be one of the most powerful procedures of quantization of gauge theories. An important feature of this approach is that, for the case of reducible gauge theories, it furnishes a systematic way of building up the non-trivial ghost

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for ghost structure. Another important advantage is that quantum corrections to the path integral measure can be calculated as long as a regularization procedure is introduced. In this framework, gauge anomalies show up as violations in the so-called quantum master equation. The successful application of the Pauli–Villars regularization scheme to this formalism at the one-loop level [6] leads to a series of results on the calculation of gauge anomalies, as well as on the implementation of the Wess–Zumino mechanism of restoring gauge invariance in the field–antifield formalism [5, 7–9].

The calculation of anomalous violations in the conservation of global currents in the field–antifield framework was discussed recently in [10, 11]. These articles consider models where there is no gauge anomaly but just anomalous divergences of some global currents. There the field–antifield formalism is extended by trivially gauging the global symmetries by means of the introduction of compensating fields in such a way that global anomalies can naturally arise from the generating functional. Different aspects of rigid symmetries inside the field antifield formalism have been discussed in [12, 13]. Here we will consider a specific model where there are in general quantum corrections to both gauge and global symmetries. This model describes a modified two-dimensional (2D) electrodynamics where arbitrary linear axial and vector current couplings were introduced and it generalizes that one introduced in [14]. Depending on the parameters chosen, the master equation at one-loop order may have no local solution in the original space of fields and antifields, which then has to be properly extended, following the ideas introduced in [15]. An interesting feature of our approach is that we can show that it is possible to calculate the anomalous divergences by means of a canonical transformation and not by introducing compensating fields in order to extend the symmetry content of the original classical theory [10, 11]. This procedure permits to extract global anomalies even for models where the gauge symmetries have been restored with the aid of Wess–Zumino fields.

Let us consider the general two dimensional Abelian gauge model, depending on two arbitrary parameters. This is more general than the theory of [14] in the sense that it includes also the possibility of pure axial coupling. Its action is given by†

$$\mathcal{S}_0[A,\psi,\bar{\psi}] = \int \mathrm{d}^2x\,\left[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathrm{i}\bar{\psi}\tilde{\not{D}}\psi\right] \tag{1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2}$$

$$\tilde{\not{D}} = \gamma^\mu\left(\partial_\mu - \mathrm{i}e(s + r\gamma_5)A_\mu\right). \tag{3}$$

The action (1) reduces to the Schwinger model when $s = 1$, $r = 0$; to 2D axial electrodynamics when $s = 0$, $r = 1$ and to the chiral Schwinger model when $s = \frac{1}{2}$, $r = \pm\frac{1}{2}$, but we pose no initial restrictions on the range of variation of these parameters. In this sense it describes any interpolation between vector and axial 2D electrodynamics, displaying in this way an interesting quantum dynamics. The model is classically invariant by the local infinitesimal transformations

$$\begin{aligned} \delta\psi &= \mathrm{i}e\alpha(x)\,(s + r\gamma_5)\,\psi \\ \delta\bar{\psi} &= -\mathrm{i}e\alpha(x)\bar{\psi}\,(s - r\gamma_5) \\ \delta A_\mu &= \partial_\mu\alpha(x) \end{aligned} \tag{4}$$

and also by the global infinitesimal transformations

$$\begin{aligned} \delta\psi &= \mathrm{i}e(\varepsilon^1 + \varepsilon^2\gamma_5)\psi \\ \delta\bar{\psi} &= -\mathrm{i}e\bar{\psi}(\varepsilon^1 - \varepsilon^2\gamma_5) \end{aligned} \tag{5}$$

† We are using: $\eta_{\mu\nu} = \mathrm{diag}(+1,-1)$, $\epsilon^{01} = +1$, $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$, $\gamma_5 = \gamma^0\gamma^1$; $\gamma^\mu = \gamma^0\gamma^{\mu\dagger}\gamma^0$, $\gamma^\mu\gamma^\nu = \eta^{\mu\nu} + \epsilon^{\mu\nu}\gamma_5$ and $\gamma_5\gamma^\mu = \epsilon^{\mu\nu}\gamma_\nu$.

where the parameters ε^1 and ε^2 are constant. It is trivial to verify that the transformations (4) close in an Abelian algebra. The Noether currents associated to the global transformations (5), with parameters ε^1 and ε^2, are respectively given by

$$\begin{aligned} j_V^\mu &= \bar{\psi}\gamma^\mu\psi \\ j_A^\mu &= \bar{\psi}\gamma_5\gamma^\mu\psi \end{aligned} \tag{6}$$

and at the classical level the Noether theorem asserts that $\partial_\mu j_V^\mu = 0 = \partial_\mu j_A^\mu$.

Now let us quantize this theory. In the field–antifield formalism the generating functional has the general form†

$$Z_\Psi[\eta] = \int [\mathrm{d}\Phi^A] \exp \frac{\mathrm{i}}{\hbar} \left\{ W\left[\Phi^A, \Phi_A^* = \frac{\partial\Psi}{\partial\Phi^A}\right] + \eta_A \Phi^A \right\} \tag{7}$$

where the quantum action is expanded in orders of $\hbar$ as $W = \mathcal{S} + \sum_{n\geqslant 1} \hbar^p M_p$ and must be a proper solution of the quantum master equation [3–5]

$$\tfrac{1}{2}(W, W) - \mathrm{i}\hbar\Delta W = 0. \tag{8}$$

This equation is formally equivalent to the independence of (7) with respect to the gauge fixing fermion Ψ. In (7) Φ^A is a set that includes the classical fields, ghosts, antighosts, auxiliary fields, etc and Φ_A^* are the corresponding antifields. The antibracket (,) and the Δ operator appearing in (8) are defined through

$$(X, Y) = \frac{\partial^R X}{\partial\Phi^A}\frac{\partial^L Y}{\partial\Phi_A^*} - \frac{\partial^R X}{\partial\Phi_*^A}\frac{\partial^L Y}{\partial\Phi_A} \tag{9}$$

$$\Delta X = (-1)^{\epsilon_A+1}\frac{\partial^R\partial^R}{\partial\Phi^A\partial\Phi_A^*}X \tag{10}$$

for arbitrary functions X and Y of the fields and antifields. The corresponding classical action is $\mathcal{S}_0 \equiv \mathcal{S}[\Phi^A, \Phi_A^* = 0]$.

Going back to our action (1), corresponding to the local symmetry (4) we introduce the ghost c and, as usual, an auxiliary trivial pair $\bar{c}$, b and write down the BV action as

$$\mathcal{S} = \mathcal{S}_0 + \int \mathrm{d}^2x \left[A^{*\mu}\partial_\mu c - \mathrm{i}e\psi^*\big[(s + r\gamma_5)c\big]\psi + \mathrm{i}e\bar{\psi}\big[(s - r\gamma_5)c\big]\bar{\psi}^* + \bar{c}^* b\right]. \tag{11}$$

In order to solve the master equation (8) at one-loop level we must calculate $\Delta\mathcal{S}$. In the field–antifield formalism, at this loop order, the Pauli–Villars (PV) regularization prescription can easily be adopted. We add to the action (11) a PV action with fields $\bar{\chi}$ and χ and choose, as usual, a mass term of the form $\mathcal{M}\bar{\chi}\chi$ that considers the vector symmetry as a privileged one. Then following the steps described in [5, 6], we find that

$$\Delta\mathcal{S} = \frac{\mathrm{i}e^2}{\pi}\int \mathrm{d}^2x\, rc\big(r\partial_\mu A^\mu + s\epsilon^{\mu\nu}\partial_\mu A_\nu\big). \tag{12}$$

The term $\epsilon^{\mu\nu}\partial_\mu A_\nu$ is cohomologically non-trivial while the term $\partial_\mu A^\mu$ is trivial [4]. If $r = 0$, $\Delta\mathcal{S}$ vanishes identically, which is the expected value, since according to (3) we would in this case have a pure vector current coupling with the gauge connection and we are adopting a regularization that takes the vector symmetry as a preferred one. If $r \neq 0$ but $s = 0$ there is no genuine anomaly in the theory as we easily see that

$$M_1 = \frac{e^2 r^2}{2\pi}\int \mathrm{d}^2x\, (A_\mu A^\mu) \tag{13}$$

† We are using de Witt's condensed notation which subtends an integral over space–time variables when pertinent. Explicitly $\eta_A\Phi^A \equiv \int \mathrm{d}^D x\, \eta_A(x)\Phi^A(x)$.

solves the master equation. This result in some sense could be expected due to the 2D identity $\gamma_5\gamma^\mu = \epsilon^{\mu\nu}\gamma_\nu$. It implies that the axial current coupling term corresponds to a vector current coupling with a modified 'gauge connection' given by $\tilde{A}_\mu = \epsilon_{\mu\nu}A^\nu$. The anomaly under an axial gauge transformation (see equation (4)). should then be proportional to $\epsilon_{\mu\nu}\tilde{F}^{\mu\nu}$, where $\tilde{F}_{\mu\nu}$ is the 'curvature' associated to $\tilde{A}_\mu$. However, $\epsilon_{\mu\nu}\tilde{F}^{\mu\nu} = 2\partial_\mu A^\mu$, which is just the cohomological trivial term found in the present situation. Of course this is not valid for higher dimensions.

In the general case, however, when $r \neq 0$ and $s \neq 0$, no M_1 can be found (in the original space of fields) which cancels ΔS in the first loop order term of the expansion of (8). Thus we are in presence of a true gauge anomalous theory. If we take the point of view of [15] and enlarge the set of fields of the theory by means of the introduction of a Wess–Zumino field θ that transforms exactly in the original gauge group as $\delta\theta(x) = \alpha(x)$, we can write a solution to the master equation as

$$M_1 = \frac{\mathrm{i}e^2}{\pi}\int \mathrm{d}^2x\ \left\{\tfrac{1}{2}ar^2A_\mu A^\mu + \tfrac{1}{2}r^2(a-1)\partial_\mu\theta\partial^\mu\theta + \theta\left[r^2(a-1)\partial_\mu A^\mu - rs\epsilon^{\mu\nu}\partial_\mu A_\nu\right]\right\}. \tag{14}$$

Since $\delta M_1 - \mathrm{i}\Delta\mathcal{S} = 0$, it follows that (8) is satisfied for

$$W = \mathcal{S} + \hbar M_1 + \int \mathrm{d}^2\, x\theta^* c. \tag{15}$$

This kind of procedure was proposed in the field–antifield framework in [8, 9]. The arbitrary parameter a which appears in (15) is to be identified with the Jackiw–Rajaraman parameter [16].

Let us now look at the global symmetries. First let us show how can we calculate the anomalous divergences of generic global currents by canonical transformations. If we perform an infinitesimal transformation generated by

$$F(\Phi^A, \Phi^{*'}_A) = \Phi^{*'}_A\Phi^A + f(\Phi^A, \Phi^{*'}_A) \tag{16}$$

with

$$f = \epsilon^\alpha\Phi^{*'}_A G^A_\alpha[\Phi] \tag{17}$$

where the ϵ^α are some arbitrary infinitesimal local parameters, the fields Φ^A, Φ^*_A transform as

$$\Phi^{A'} = \frac{\partial^L F}{\partial\Phi^{*'}_A} = \Phi^A + \epsilon^\alpha G^A_\alpha[\Phi] \tag{18}$$

$$\Phi^*_A = \frac{\partial^R F}{\partial\Phi^A} \quad\to\quad \Phi^{*'}_A = \Phi^*_B\left(\delta^B_A - \epsilon^\alpha\frac{\partial^R G^B_\alpha}{\partial\Phi^A}\right). \tag{19}$$

It is possible to show [5, 6] that the new action

$$\tilde{W} = W' - \mathrm{i}\hbar\Delta f \tag{20}$$

where W' is the action W written in terms of the new primed variables, also satisfies (8).

The factor Δf compensates the possible non-invariance of the measure of integration of (7) with respect to the canonical transformation. However, exactly as happens in the case of ΔS, the action of the operator Δ must be regularized. When the regularized result is non-vanishing, it can be put in the form

$$-\mathrm{i}\Delta f = \epsilon^\alpha\mathcal{A}_\alpha[\phi^i]. \tag{21}$$

A generating functional $Z_\Psi[J,\epsilon]$ constructed with $\tilde{W}$ naturally leads to the same quantized theory and since the parameters ϵ^α are arbitrary, we must have the identities

$$\frac{\partial Z_\Psi[J,\epsilon]}{\partial\epsilon^\alpha(x)} \equiv 0. \tag{22}$$

Whenever $\Delta f \neq 0$ these identities may be used to derive anomalous results involving $\mathcal{A}_\alpha$. Introducing

$$j^\mu_\alpha = \frac{\partial W'}{\partial(\partial_\mu \epsilon^\alpha)} \tag{23}$$

we get the general relation involving the quantum expectation value of the divergence of this current:

$$\frac{\partial Z_\Psi[J,\epsilon]}{\partial \epsilon^\alpha(x)} = \frac{\mathrm{i}}{\hbar}\left\langle -\partial_\mu j^\mu_\alpha + \frac{\partial W'}{\partial \epsilon^\alpha} - \hbar \mathcal{A}_\alpha - G^A_\alpha[\Phi]\eta_A \right\rangle. \tag{24}$$

Therefore, to calculate the anomalous contributions to the divergence of our global currents (6) we perform a canonical transformation generated by (16) with

$$f = -\mathrm{i}\epsilon^1\big(\psi^{*'}\psi - \bar\psi\bar\psi^{*'}\big) - \mathrm{i}\epsilon^2\big(\psi^{*'}\gamma_5\psi + \bar\psi\gamma_5\bar\psi^{*'}\big) \tag{25}$$

that encompasses both the vectorial and axial transformations

$$\begin{aligned}
\psi' &= \big(1 - \mathrm{i}\epsilon^1 - \mathrm{i}\epsilon^2\gamma_5\big)\psi \\
\bar\psi' &= \bar\psi\big(1 + \mathrm{i}\epsilon^1 - \mathrm{i}\epsilon^2\gamma_5\big) \\
\psi^{*\prime} &= \psi^*\big(1 + \mathrm{i}\epsilon^1 + \mathrm{i}\epsilon^2\gamma_5\big) \\
\bar\psi^{*\prime} &= \big(1 - \mathrm{i}\epsilon^1 + \mathrm{i}\epsilon^2\gamma_5\big)\bar\psi^*
\end{aligned} \tag{26}$$

keeping the remaining fields unchanged.

Assuming a regularization analogous to that one adopted for the calculation of (12), the derivation of Δf is analogous to that of ΔS with rc replaced by ϵ. This gives

$$\Delta f = \frac{\mathrm{i}e^2}{\pi}\int \mathrm{d}^2x\, \epsilon\big(r\partial_\mu A^\mu + s\epsilon^{\mu\nu}\partial_\mu A_\nu\big). \tag{27}$$

We will assume that, as usual, the gauge fixing fermion Ψ in (7) does not depend on the fermionic fields. This is equivalent to setting

$$\bar\psi^* = \psi^* = 0 \tag{28}$$

at the gauge-fixed level (after the canonical transformations are performed).

Substitution of (26) and (27) in (20) leads through (22) to

$$\begin{aligned}
\langle \partial_\mu j^\mu_A \rangle\big|_{\eta=\bar\eta=0} &= \frac{\hbar e^2}{\pi}\left\langle \int \mathrm{d}^2x\, \big(r\partial_\mu A^\mu + s\epsilon^{\mu\nu}\partial_\mu A_\nu\big)\right\rangle \\
\langle \partial_\mu j^\mu_V \rangle\big|_{\eta=\bar\eta=0} &= 0
\end{aligned} \tag{29}$$

where η and $\bar\eta$ are the external sources corresponding to ψ and $\bar\psi$, respectively.

These results, together with equation (12), show us that our model has, in general, both gauge and global anomalies (violation of the conservation of global currents). The fact that the vector current is always conserved clearly reflects the choice of a mass term for the Pauli–Villars regulating fields that preserves the vector symmetry. It is interesting to observe that the axial current will always have an anomalous divergence even in the particular cases where there is no gauge anomaly.

It is interesting to see what happens in the non-Abelian case. The non-Abelian version of (1) reads

$$\mathcal{S}_0\big[A,\psi,\bar\psi\big] = \int \mathrm{d}^2x\,\big[-\tfrac{1}{4}\,\mathrm{tr}\,F^{\mu\nu}F_{\mu\nu} + \mathrm{i}\bar\psi\tilde{\not{D}}\psi\big] \tag{30}$$

where we now have

$$\begin{aligned}
&\tilde{\not{D}} = \gamma^\mu\big(\partial_\mu - \mathrm{i}e(s + r\gamma_5)A_\mu\big)\\
&F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}e[A_\mu, A_\nu]\\
&A_\mu = A_\mu^a T^a\\
&\big[T^a, T^b\big] = \mathrm{i} f^{abc} T^c\\
&\mathrm{tr}\, T^a T^b = \delta^{ab}.
\end{aligned} \tag{31}$$

The natural generalization of the transformations (4) reads

$$\begin{aligned}
&\delta\psi = \mathrm{i}e(s + r\gamma_5)\alpha\psi\\
&\delta\bar{\psi} = -\mathrm{i}e\bar{\psi}(s - r\gamma_5)\alpha\\
&\delta A_\mu = \partial_\mu\alpha - \mathrm{i}e[A_\mu, \alpha]
\end{aligned} \tag{32}$$

where the infinitesimal parameters $\alpha \equiv \alpha^a T^a$ now take values of the Lie algebra of the group generated by T^a. However, a simple calculation shows that these transformations leave the action (30) invariant if and only if

$$(s + r\gamma_5)^2 = (s + r\gamma_5). \tag{33}$$

This restricts the range of variations of the pair (s, r) to $(1, 0)$ or $(\frac{1}{2}, \pm\frac{1}{2})$ (besides the trivial $(0, 0)$). So, in the non-Abelian case, the action (30) does not represent any new general model, but only a concise representation for chiral or vector QCD_2. There is an extensive body of literature about these two models [17], to which we refer the reader. It would be possible, however, to apply the methods described here, as well as those found in [10, 11], to any of these two non-Abelian models.

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References

[1] Adler S 1969 *Phys. Rev.* **177** 2426
Bell J and Jackiw R 1969 *Nuovo Cim.* A **60** 47
[2] For a review, see: Jackiw R 1972 *Lectures on Current Algebra and its Applications* ed S Treiman *et al* (Princeton, NJ: Princeton University Press)
[3] Batalin I A and Vilkovisky G A 1981 *Phys. Lett.* B **102** 27
Batalin I A and Vilkovisky G A 1983 *Phys. Rev.* D **28** 2567
[4] For a review, see Henneaux M and Teitelboim C 1992 *Quantization of Gauge Systems* (Princeton, NJ: Princeton University Press)
[5] A recent review with a wide list of references can be found in: Gomis J, Paris J and Samuel S 1995 *Phys. Rep.* **259** 1
[6] Troost W, van Nieuwenhuizen P and Van Proeyen A 1990 *Nucl. Phys.* B **333** 727
[7] Braga N F R and Montani H 1991 *Phys. Lett.* B **264** 125
[8] Braga N F R and Montani H 1993 *Int. J. Mod. Phys.* A **8** 2569
[9] Gomis J and Paris J 1993 *Nucl. Phys.* B **395** 288
[10] Amorim R and Braga N R F 1998 *Phys. Rev.* D **57** 1225
[11] Amorim R, Braga N R F and Henneaux M 1998 *Phys. Lett.* B **436** 125
[12] Brandt F, Henneaux M and Wilch A 1998 *Nucl. Phys.* B **510** 640
[13] Hurth T and Skenderis K 1998 Quantum Noether method *Preprint* hep-th/9803030 (*Nucl. Phys.* B to appear)
[14] Bassetto A, Griguolo L and Zanca P 1994 *Phys. Rev.* D **50** 1077
[15] Faddeev L D 1984 *Phys. Lett.* B **145** 81
Faddeev L D and Shatashvili S L 1986 *Phys. Lett.* B **167** 225
[16] Jackiw R and Rajaraman R 1985 *Phys. Rev. Lett.* **54** 1219
[17] See, for instance: Abdalla E and Abdalla M C B 1996 *Phys. Rept.* **265** 253 and references therein